Infrastructure Topology Optimization under Competition through Cross-Entropy

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Abstract

In this article, we study a two-level non-cooperative game between providers acting on the same geographic area. Each provider has the opportunity to set up a network of stations so as to capture as many consumers as possible. Its deployment being costly, the provider has to optimize both the number of settled stations as well as their locations. In the first level each provider optimizes independently his infrastructure topology while in the second level they price dynamically the access to their network of stations. The consumers' choices depend on the perception (in terms of price, congestion and distances to the nearest stations) that they have of the service proposed by each provider. Each provider market share is then obtained as the solution of a fixed point equation since the congestion level is supposed to depend on the market share of the provider which increases with the number of consumers choosing the same provider. We prove that the two-stage game admits a unique equilibrium in price at any time instant. An algorithm based on the cross-entropy method is proposed to optimize the providers' infrastructure topology and it is tested on numerical examples providing economic interpretations.

Keywords: Non-cooperative game; Implicit function; Cross-entropy method

1 Introduction

Global warming appears as one of the major concerns of the governments, today. Measures to reduce the greenhouse effect take various forms like the limitations of the right to pollute through the introduction of taxes leading to the emergence of CO_2 markets where rights to pollute are negotiated between the involved partners, the development of more intelligent infrastructures also called smart grids leading to the optimization of the energy consumption, the introduction on the market of Electric Vehicles (EVs), etc.

This latter objective is quite controversial because the adoption of the EVs depends deeply on the perception of the consumers and in particular on their degree of concern

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regarding the autonomy of the vehicle. In turn, the EV range of autonomy is conditional on the latest industrial discoveries about the manufacture of batteries, especially their chemical constitutions [22].

Great lobbying efforts to promote green technologies and change the consumption patterns are required to guarantee the EV adoption. But the supports of the various governments are correlated to their will to become energy independent while providing new sources of economic growth [22]. Additionally, their involvement is necessary to give clear guidelines about the use of the EVs: Will they be reserved for intra-urban usage with a monthly renewable subscription imposed to rent one vehicle of the city maintained fleet or will their usage be extended to personal use involving longer distance travels? Which reloading process will be privileged: battery switching where once discharged the batteries are quickly changed in stations or a mix of short charge reloadings in public stations located along the road and long charges at home?

From a technical point of view, the introduction of the EVs requires to address three areas of research:

- First, we need to develop efficient prediction techniques in order to forecast the energy demand in the stations. A first approach has been proposed in [16] where the problem is modeled as a partial information game and a machine learning approach based on regret minimization is provided to forecast online the energy demand.

- Second, it is necessary to apply planning techniques in order to optimize the charging ordering. Various techniques of operations research issued from the management of the supply chain can be applied [30]. Dynamic programming can also represent an alternative. But it is limited by the curse of dimensionality especially if the number of states representing for instance the number of clients entering simultaneously the charge station, is large [28].

- Third, the charging infrastructure topology needs to be optimized. It requires to determine simultaneously the optimal number of charge stations as well as their locations. Additionally, the setting up of a station being costly, the infrastructure topology optimization is inseparable from the underlying economic concerns such as: Who invest in the infrastructure? What incentives can push the providers to invest?

In this article, we focus on the third point.

Technically, the optimization of the charging infrastructure topology belongs to the category of problems dealing with *facility location*. Facility location is a branch of operations research and computational geometry concerned itself with mathematical modeling and proposing solutions to optimize the placement of facilities in order to minimize transportation cost, outperform competitors' facilities, etc. It represents some of the most widely studied problems in combinatorial optimization [12]. In the basic formulation, a set of demand points being fixed, the objective is to pick a subset of the set containing all the facilities to open to minimize the sum of the distances from each demand point to its nearest facility plus the sum of opening costs of the facilities.

The literature dealing with facility location can be divided into 4 categories: p-median problems, p-center problems, uncapacitated facility location problems (UFLP), capacitated facility location problems (FLP) [12], [26].

We now give some bibliographic details about these 4 categories. p-median problems find medians among existing weigthed points corresponding to the demand points on a graph. Taking as applicative starting point the optimization of the locations of switching centers in a communication network, Hakimi shows that it is always possible to find a collection of p optimal sites for the facility settings at vertices of the graph [26]. In p-center problems, the goal is to minimize the maximum distance between points and centers. A classical illustration for $p = 1$ is the Fermat-Weber's problem where the objective is to place a single facility so as to minimize the sum of the distances from a given set of points [26]. In UFLP, the objective is to choose sites among a set of candidates in which facilities can be located so that the demands of a given set of clients are satisfied at minimum costs. Besides, the capacities of all the facilities are infinite. Unlike p-median problems, UFLP does not impose any constraint on the maximum number of facilities and a cost is associated to the location of a facility making the link with economics. Finally, FLP problems are similar to UFLP but a capacity constraint is imposed on each facility.

The facility location problem on general graphs is NP-hard to solve optimally [12], [26]. A number of approximation algorithms have been developed. Tentatives are based on heuristics (such as greedy or alternate algorithms, vertex substitution, etc.), metaheuristics (composed of various search approaches in graphs such as variable neighborhood search, scatter search, etc., heuristic concentration, genetic algorithms, tabu search, simulated annealing, branch and bound being the most widely used), approximation algorithms, linear programming relaxation, integer programming formulations and reductions, enumerations, etc. Reese presents a survey of the algorithmic techniques and provides detailed bibliographic references in [26]. Many research articles concentrate now on combinations of the above mentioned algorithms in order to optimize the convergence speed and accuracy. For instance, Gosh checks that the combination of tabu search and complete local search with memory enables the solving of large instances of UFLP and outperforms other simple approximations [10].

Facility location problems are strongly linked to economics, more specifically the design of supply chain structures and algorithmic game theory. To give a few up to date illustrations, Albareda-Sambola et al. consider multi period location problems where at each time instant the decision maker decides which existing facilities should be closed and where new facilities should be opened [1]. Dürr and Nguyen tackle the problem of placing facilities on the nodes of a metric network inhabited by a fixed number of autonomous self-interested agents [9], [32]. They provide bounds for the strategyproof mechanisms where none of the players can be better off by misreporting his facility locations. The agents' choices and therefore the optimal facility locations are based solely on the distance between the agents and the available facilities [9], [32]. Borndörfer et al. consider the problem of optimizing the spatial distribution of controls over a motorway in order to maximize the sum of the revenues generated by transit fees imposed to the drivers assuming evasion is possible [5]. It is solved as a Stackelberg game using linear and mixed integer programs. For the first time in the literature, the impact of the network topology and of the spatial distribution of the controls are taken into account in the game solution. However, it requires to make a number of simplifying assumptions regarding the drivers: they are route takers implying that their decision reduces to pay or evade and there is no variation of sensitivity to the penalties between them.

This article is placed in a context of competition between service providers playing in the same geographic area. Each provider manages a network of stations called charging infrastructure. He determines the number of stations to be settled and their locations on the plane. Once the charging infrastructure has been settled, the provider has to determine dynamically the access price to his stations over a finite time horizon. The access price is the price that the EV drivers pay to charge their vehicle in the station. It is updated at each time instant of the second level game. We aim at answering the following questions:

- Is it possible to determine the charging infrastructure topology in terms of size and locations, maximizing the provider's revenue?
- What are the optimal dynamic pricing strategies under competition between the providers?

The originality of the article lies in two main aspects. First, compared to classical approaches used to model consumers' choices [14], [15], [18], [23], [25], we have chosen to incorporate the congestion level usually modeled as quality of service or time before service, in the EV drivers choices. This complexity makes the game resolution harder than under traditional approaches where the congestion level is supposed fixed or depending exclusively on the service providers' investments. Second, the use of stochastic optimization enables us to solve large instances of the optimal design of the charging infrastructure problem under dynamic pricing.

The article is organized as follows. In Section 2, we describe the EV drivers' choice model and the two-stage game between the service providers. In Section 3, the pricing game to determine the service providers' optimal access prices to their stations is solved. In Section 4, a simulation approach based on the cross-entropy method is proposed to optimize the providers' charging infrastructure topology. Finally, in Section 5, numerical illustrations highlight the potential of the elaborated method to make decisions in an uncertain and competitive context while providing economic guidelines.

To facilitate the understanding, the main notations used throughout the article are summarized in the table below.

S_k	Service provider k
E	Energy provider
N_k	Number of stations settled by provider S_k
$s_k(l)$	Coordinates of provider S_k's l-th station

s_k	Vector containing the coordinates of all the stations managed by provider S_k
$\theta_k(t)$	Provider S_k market share at t
$p_k(t)$	Provider S_k access price at t
I_k	Provider S_k unit investment level
$p_E(t)$	Energy provider unit selling price
$N(t)$	Number of Electric Vehicles wishing to reload at t
$\pi_k(t)$	Provider S_k's utility
c_E	Energy provider's investment cost in the grid
$\bar{d}_k(t)$	Mean value of the distance between the EV drivers and provider S_k's stations
$v(., Q(t))$	Density generating the EV locations on the $\mathbb{R}^2$ plane
$Q(t)$	Dynamic parameter of the above density at t
$c_l(k, t)$	Opportunity cost of EV l associated to provider S_k's service at t
β_l	Congestion sensitivity coefficient of EV l
$q_k(t)$	Provider S_k's congestion level at t
T	Finite horizon of the repeated game
$\varphi(., .)$	Function describing the congestion
$B_{1,2}(t)$	Indifference bound between S_1 and S_2 at t
$B_{k,0}(t)$	Indifference bound between S_k and no reload at t
$c_{\max}$	Maximum admissible opportunity cost
i_k	Extended capacity of provider S_k
μ_k	Provider S_k's capacity
δ	Discount factor
$f(., \rho_k)$	Density generating provider S_k's station locations
ρ_k	Parameter of the above density
M	Number of iterations of the cross-entropy algorithm
$\mathcal{N}_s$	Sample size
ζ	One minus the value of the quantile associated to the performance statistics
d	Stopping criterion

2 The model

In this section, we describe the involved players, their economic relationships and give a mathematical formulation for their utilities.

We consider two competitive service providers S_1 and S_2. Each provider manages a fixed number of stations: N_1 for provider S_1 and N_2 for provider S_2 with $N_1, N_2 > 0$. The vector containing the stations of provider $S_k,\ k = 1, 2$ locations is denoted: $s_k = (s_k(1), ..., s_k(N_k))$ where $s_k(l) \in \mathbb{R}^2$ contains the coordinates of provider S_k's l-th station. At any time instant, EV drivers needing to reload can enter one of the service providers' charge stations or delay their reload. Their choice depends on their intrinsic preferences on prices, on the congestion levels in the charge stations and on the distances to the providers' nearest stations[1]. At time instant t, we let $\theta_k(t) \in [0; 1]$ be provider S_k's market share. $N(t) \in \mathbb{N}$ is the dynamic process containing the total number of EV drivers needing to reload at time instant t.

[1]It is assumed that online informations regarding the total number of EV drivers wishing to reload, prices, congestion levels and distance to stations are publicly accessible both for the EVs, through electronic technologies like smart phones for instance, and for the service providers.

In a first step, we describe provider $S_k,\ k = 1, 2$ utility. Provider $S_k,\ k = 1, 2$ determines the access to his station at a price $p_k(t)$. This is the fee required to park and make one's battery change/car reload, once in the station. Additionally, investing is necessary to decrease the congestion level in the charge stations and more generally, to improve the perceived quality. Since at the beginning of each time period, the service provider ignores how the EV drivers will distribute among them, he is forced to make investments considering all the EV drivers needing to reload on the plane over this period i.e., $N(t)$. $I_k \in \mathbb{R}$ captures the speed of investment per EV driver of provider S_k. The higher is I_k, the higher is provider S_k's investment in the infrastructure for $k = 1, 2$. The service providers buy energy from an energy provider called E who is supposed to be in monopoly, at a unit price $p_E(t)$ fixed by E. A specificity of the energy is that it is very difficult to store. An envisaged possibility might be to build energy stocks in the EV batteries, for instance at night [22]. The development of efficient planning algorithms for the optimization of the (incoming and outcoming) energy flows in virtual centrals constitutes one of the major challenges associated with the smart grid management. However, the case of virtual centrals is out of the scope of this article. It will be the subject of another article focusing on stochastic games. Therefore, we assume that the service providers buy the quantity of energy coinciding exactly with his clients demand. Using the previous definitions, the utility of any provider $S_k,\ k = 1, 2$ is defined as the difference between the revenue generated by the EV drivers choosing to reload in his stations and the cost to buy energy on provider E's grid so as to satisfy the EV drivers' demand minus the investment which is necessary to decrease the congestion level:

$$\pi_k(t) = \Big((p_k(t) - p_E(t))\theta_k(t) - I_k\Big)N(t),\ \forall k = 1, 2 \tag{1}$$

In a second step, we detail the energy provider's utlity. The energy provider E receives energy from energy producers using either non-renewable or renewable sources. Let $c_E \geq 0$ be the fixed cost representing energy provider E investment in the grid per EV driver. Note that the energy provider's investment cost can take various forms. In general, E has to manage a public infrastructure and the investment is financed mostly from the citizens' monthly taxes. The energy provider E's utility is obtained as the difference between the revenue generated from the service providers who buy energy on his grid and his investment to prevent congestion and degradation on his grid:

$$\pi_E(t) = \Big(p_E(t) \underbrace{\sum_{k=1,2} \theta_k(t)}_{\text{Total market share of the providers}} - c_E\Big)N(t) \tag{2}$$

In all the article, the distance $d(.,.)$ between two vectors of the plane coincides with the L^2- norm. Any EV driver l is located on the plane by a position vector modeled as a random vector $X(t) \in \mathbb{R}^2$. We give justifications about this statistical assumption: Provider $S_k,\ k = 1, 2$ does not observe the individual position of the EV drivers. However, he collects enough information to infer the density function corresponding to the distribution of their position on the plane. This density function is supposed constant over time interval $[t; t+1[$ for any $t > 0$. As a result, we assume that the EV driver locations on time interval $[t; t+1[$ is a random variable distributed according to a density function $v(., Q(t))$ whose exogeneous parameter $Q(t)$ is time-dependent. For instance, we might assume that the EV drivers' trajectory is distributed according to a two-dimensional brownian motion with correlated components. Section 5 provides an illustration of this assumption. At time instant t, the mean value of the distance between the EV drivers and provider S_k stations is obtained as the expectation of the minimum distance between S_k stations locations and the EV drivers locations:

$$\begin{aligned} \bar{d}_k(t) &= \mathbb{E}_{X(t)}\Big[\min_{l=1,\dots,N_k} d(X(t), s_k(l))\Big] \\ &= \int_{\mathbb{R}^2} \min_{l=1,\dots,N_k} d(x, s_k(l)) v(x, Q(t)) dx \end{aligned} \tag{3}$$

An EV driver wishing to reload starts by computing the opportunity cost associated with each service provider. The notion of opportunity cost is classical in economics [3], [14], [15], [25]. It is used to model the interdependences between the consumers preferences regarding various attributes. Here, the attributes coincide with prices, congestion levels and distances to the nearest stations. Opportunity costs are used in social sciences and telecommunication economics to determine the providers' market shares in case of oligopolies [3], [14], [15]. We suppose that the EV driver chooses the service provider having the smallest opportunity cost or delay his reload in case where all the opportunity costs are superior to his maximum admissible opportunity cost, $0 < c_{\max} < +\infty$. This maximum admissible opportunity cost is identical for all the EV drivers. At time instant t, the congestion in provider S_k's stations $q_k(t)$, is measured by the mean waiting time. It is captured by a function $\varphi(.)$ of provider S_k's market share and investment. $\varphi(.)$ is $\mathcal{C}^2$ i.e., continuous, twice differentiable and of continuous differentiates, both in $\theta_k(t)$ and in I_k. Besides, it is increasing in $\theta_k(t)$ but decreasing in I_k. Under all these assumptions, EV driver l's opportunity cost toward provider S_k is of the form:

$$\begin{aligned} c_l(k,t) &= p_k(t) + \beta_l q_k(t) + \bar{d}_k(t) \\ q_k(t) &= \varphi(\theta_k(t), I_k) \end{aligned} \tag{4}$$

where $\beta_l \in [0; 1]$ is a coefficient modeling EV driver l's sensitivity to the congestion in the station. Having no a priori information about the EV drivers' preferences, we assume that the congestion sensitivity coefficient is distributed according to the uniform density on the interval $[0; 1]$ i.e., $\beta_l \sim \mathcal{U}[0; 1], \ \forall l = 1, ..., N(t)$[2].

Station locations being fixed, the problem can be modeled as a *repeated game with complete information* and finite horizon $T < +\infty$ since at each time instant, each service provider wants to maximize selfishly his utility by selecting his access price [21], [34]. However, the service provider's access price optimization depends on the

[2]The analytical results obtained in this article can be extended to the case where we choose another density such as normal, truncated exponential, of gamma type, khi-deux, etc. However, under such density assumptions, it might not be possible to derive the analytical expressions of the service providers' market shares and a numerical approach should be envisaged.

EV drivers' choices. The station locations chosen initially by the service providers is fundamental since it influences directly the EV drivers' choice to reload and in turn, the service providers' pricing strategies. Formally, the game can be described as two stages with timing delay between the determination of station locations and station access pricing:

(1) At $t = 0$, each service provider $S_k,\ k = 1, 2$ determines his station locations simultaneously and independently. The station coordinates are then publicly announced.

For $t = 1, ..., T$, a pricing game is played repeatedly

(2) Each service provider S_k determines simultaneously and independently the station access prices maximizing his utility $\pi_k(t),\ k = 1, 2$.

(3) The EV drivers choose their service provider or delay their reload.

The repeated pricing game (steps (2) and (3)) is solved in Section 3. Step (1) i.e., the optimal positioning of the providers' stations is considered in Section 4.

3 Pricing game resolution

In this section, we suppose that both providers have a fixed number of stations and that their locations are known publicly i.e., step (1) of the game described in Section 2 is supposed to have already been considered. In this section, the pricing game described in steps (2) and (3) is solved analytically proceeding by backward induction. In particular, we prove that it admits a unique Nash equilibrium in price associated with a unique allocation of the EV drivers between the providers.

3.1 EV drivers allocation between the providers

The congestion sensitivity coefficient being distributed according to a uniform density over interval $[0; 1]$, the analytical derivation of the providers' market shares can be obtained by decomposing interval $[0; 1]$ in sub-intervals over which the EV drivers preferences are homogeneous. These sub-interval bounds correspond to the case where the EV drivers are indifferent between both providers and between the providers and the possibility to delay their reload.

Formally, any EV driver l is indifferent between service providers S_1 and S_2 if, and only if, the opportunity costs associated to each provider coincide i.e.: $c_l(1, t) = c_l(2, t)$. With the same reasoning, any EV driver l is indifferent between service providers $S_k,\ k = 1, 2$ and no reloading if, and only if, $c_l(k, t) = c_{\max},\ k = 1, 2$.

For the sake of simplicity, we introduce the *indifference bounds*:

- Between S_1 and S_2

$$B_{1,2}(t) \equiv \frac{\Big(p_1(t) - p_2(t)\Big) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big)}{\varphi\Big(\theta_2(t), I_2\Big) - \varphi\Big(\theta_1(t), I_1\Big)}$$

- Between provider $S_k,\ k = 1, 2$ and no reloading

$$B_{k,0}(t) \equiv \frac{c_{\max} - p_k(t) - \bar{d}_k(t)}{\varphi\Big(\theta_k(t), I_k\Big)}$$

In the following lemma, we detail how the indifference bounds characterizing the EV drivers' preferences are derived from the comparison of the opportunity costs.

Lemma 1. *At time instant t, EV driver l is indifferent between provider S_1 and provider S_2 if, and only if, $\beta_l = B_{1,2}(t)$. He is indifferent between no reload and S_1 (resp. S_2) if, and only if, $\beta_l = B_{1,0}(t)$ (resp. $\beta_l = B_{2,0}(t)$).*

Proof of Lemma 1. It can be found in Appendix.

The determination of the ordering of the indifference bounds on interval $[0; 1]$ enables us to derive the analytical expressions of the providers' market shares. We prove in the following lemma that this ordering depends on the sign of the difference between the provider congestion levels.

Proposition 2. *Case* 1*: $q_2(t) < q_1(t)$. Providers S_1 and S_2's market shares are defined as*

$$\begin{aligned}
\theta_1(t) &= \frac{\Big(p_1(t) - p_2(t)\Big) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big)}{\varphi\Big(\theta_2(t), I_2\Big) - \varphi\Big(\theta_1(t), I_1\Big)} - \frac{c_{\max} - p_1(t) - \bar{d}_1(t)}{\varphi\Big(\theta_1(t), I_1\Big)} \\
\theta_2(t) &= 1 - \frac{\Big(p_1(t) - p_2(t)\Big) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big)}{\varphi\Big(\theta_2(t), I_2\Big) - \varphi\Big(\theta_1(t), I_1\Big)}
\end{aligned}$$

Case 2*: $q_1(t) < q_2(t)$. Provider S_1's market share is*

$$\theta_1(t) = 1 - \frac{\Big(p_1(t) - p_2(t)\Big) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big)}{\varphi\Big(\theta_2(t), I_2\Big) - \varphi\Big(\theta_1(t), I_1\Big)}$$

and provider S_2's market share is

$$\theta_2(t) = \frac{\Big(p_1(t) - p_2(t)\Big) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big)}{\varphi\Big(\theta_2(t), I_2\Big) - \varphi\Big(\theta_1(t), I_1\Big)} - \frac{c_{\max} - p_2(t) - \bar{d}_2(t)}{\varphi\Big(\theta_1(t), I_1\Big)}$$

Proof of Proposition 2. It can be found in Appendix.

In the rest of the article, we focus on Case 1[3] i.e., we assume that the congestion level in provider S_2's stations is strictly inferior to the one in provider S_1's stations i.e.,

[3]Case 2 can be solved similarly by reversing the role played by both service providers.

$q_2(t) < q_1(t)$. This assumption can be justified by assuming that provider S_2 invests far more than S_1 in his charging infrastructure i.e., $I_2 >> I_1$.

We introduce constraints on the station access prices guaranteeing that both providers might emerge i.e., that none of them is condemned a priori to have a zero market share. The following lemma is straightforward

Lemma 3. *At time instant* t*, both providers' prices are chosen so that* $p_2(t) - p_1(t) > \bar{d}_1(t) - \bar{d}_2(t)$*, otherwise provider* S_1 *would have a zero market share.*

Proof of Lemma 3. It can be found in Appendix.

Suppose that there exists $0 < p_{\max} < +\infty$ so that at any time, $p_1(t), p_2(t) \leq p_{\max}$. Judging by the providers' utilities as defined in Equation (1), it is necessary to impose that $p_1(t), p_2(t) \geq p_E(t)$ at any time to guarantee the non-negativity of the providers' utilities. Additionally, using Lemma 3, we infer the following price ordering:

$$p_E(t) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big) \leq p_1(t) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big) < p_2(t) \leq p_{\max} \tag{5}$$

In Proposition 2, $\theta_1(t)$ (resp. $\theta_2(t)$) is still function of $\theta_1(t)$ and $\theta_2(t)$ i.e., we have no simple expression of $\theta_2(t)$ as a function of $\theta_1(t)$ solely. However to solve as formally as possible the Stackelberg game described in Section 2, we need to express $\theta_2(t)$ as a simple function of $\theta_1(t)$.

Let

$$\begin{aligned} \mathcal{C} &\equiv \Big\{(\theta_1(t), \theta_2(t)) | \theta_1(t) + \theta_2(t) \leq 1,\ \theta_1(t), \theta_2(t) \in [0;1],\ \varphi\Big(\theta_2(t), I_2\Big) \\ &< \varphi\Big(\theta_1(t), I_1\Big)\Big\} \end{aligned}$$

be the constraint space containing all the possible market share allocations under the assumption that $q_2(t) < q_1(t)$ at any time instant t. It is convex as an intersection of convex spaces.

The two equations characterizing the providers' market shares obtained in Proposition 2 are recalled below under the constraint that the associated allocations belong to space $\mathcal{C}$

$$\mathfrak{G}(1) \begin{cases} \theta_1(t) = B_{1,2}(t) - B_{1,0}(t) \\ \theta_2(t) = 1 - B_{1,2}(t) \\ (\theta_1(t), \theta_2(t)) \in \mathcal{C} \end{cases}$$

In order to fix the ideas, we give the mathematical expression of the congestion level which is experienced over provider S_k's charging infrastructure. The chosen measure is issued from network optimization literature where the congestion of a link is traditionally evaluated as a convex latency function of the flow crossing the link [24]. We assume that the congestion level is *averaged over all the stations* managed by S_k and that it is measured as the difference between the market share ($\theta_k(t)$) and the normalized (extended) capacity level. Indeed, provider S_k has the opportunity to invest in his infrastructure to reduce the congestion in his charge stations. The effect of this investment on the extension of the charge station capacities ($\mu_k \in [0;1]$) is modeled thanks to the function $\tilde{\varphi}(.)$ which is twice differentiable, non-negative, continuous and increasing in the provider's investment I_k. More precisely, the formal expression of the congestion level is:

$$\begin{aligned} q_k(t) &= \varphi\Big(\theta_k(t), I_k\Big) \\ &= \Big(\theta_k(t) - \underbrace{(\mu_k + \tilde{\varphi}(I_k))}_{i_k}\Big) \\ &= \Big(\theta_k(t) - i_k\Big). \end{aligned}$$

For the sake of simplicity, we set $i_k \equiv \mu_k + \tilde{\varphi}\Big(I_k\Big)$.

We give an interpretation of the capacity constraint on the congestion level. One originality of the model is that provider S_k has the opportunity to extend his capacity to avoid congestion.

- If $\theta_k(t) \leq i_k$ then no congestion occurs in provider S_k's charge stations. In this case $q_k(t) \leq 0$ and all the arriving EV drivers are served at provider S_k's stations i.e., there is no queue at the entries of the stations. In this case, the experienced quality appears as a measure of the speed at which the EV drivers are served.
- If $\theta_k(t) > i_k$ then $q_k(t) > 0$ and congestion is experienced in provider S_k's stations. In this case, provider S_k's station capacity is exceeded and a queue is being formed. The queue increases linearly in the number of EV drivers arriving to be served. Therefore, the risk for a queue to appear depends on provider S_k's station capacities (μ_k) and on the provider's effort to extend his station capacity ($\tilde{\varphi}(I_k)$).

Proposition 4. *Assuming that $\varphi(.,.)$ is linear in the provider's market share, we prove that at equilibrium, provider S_2's market share can be expressed exclusively as a a function of provider S_1's market share.*

Proof of Proposition 4. We detail the analytical expressions for the first and second equations of System $\mathfrak{G}(1)$. The first equation can be rewritten as:

$$\begin{aligned} &\theta_1(t)\Big(\varphi(\theta_2(t), I_2) - \varphi(\theta_1(t), I_1)\Big)\varphi(\theta_1(t), I_1) = \Big[(p_1(t) - p_2(t)) + (\bar{d}_1(t) \\ - &\bar{d}_2(t))\Big]\varphi(\theta_1(t), I_1) - \Big[c_{\max} - p_1(t) - \bar{d}_1(t)\Big]\Big[\varphi(\theta_2(t), I_2) \\ - &\varphi(\theta_1(t), I_1)\Big] \end{aligned} \quad (6)$$

For the second equation, we get:

$$\begin{aligned} &(\theta_2(t) - 1)\Big[\varphi(\theta_2(t), I_2) - \varphi(\theta_1(t), I_1)\Big] = -\Big(p_1(t) - p_2(t)\Big) - \Big(\bar{d}_1(t) \\ - &\bar{d}_2(t)\Big) \end{aligned} \quad (7)$$

By substitution of Equation (7) in Equation (6) and by simplification by $-\Big(p_1(t)-p_2(t)\Big)-\Big(\bar{d}_1(t)-\bar{d}_2(t)\Big)$, it comes:

$$\varphi(\theta_1(t), I_1)\Big[\theta_1(t)+\theta_2(t)-1\Big]+\Big[c_{\max}-p_1(t)-\bar{d}_1(t)\Big]=0$$

We then let:

$$\mathcal{G}(\theta_1(t),\theta_2(t)) \equiv \varphi(\theta_1(t), I_1)\Big[\theta_1(t)+\theta_2(t)-1\Big]+\Big[c_{\max}-p_1(t)-\bar{d}_1(t)\Big] \qquad (8)$$

Using this simplifying notation, System $\mathfrak{G}(1)$ becomes:

$$\mathfrak{G}(2)\begin{cases} \mathcal{G}(\theta_1(t),\theta_2(t))=0 \\ (\theta_1(t),\theta_2(t))\in\mathcal{C} \end{cases}$$

In System $\mathfrak{G}(2)$ we aim at describing the intersection of the zero-level set of the function $\mathcal{G}(.,.)$ with the constraint space, $\mathcal{C}$. Differentiating $\mathcal{G}(.,.)$ with respect to $\theta_1(t)$, we get $\frac{\partial\mathcal{G}(\theta_1(t),\theta_2(t))}{\partial\theta_1(t)}=\theta_1(t)+\theta_2(t)-1$. Identically, with respect to $\theta_2(t)$, we obtain $\frac{\partial\mathcal{G}(\theta_1(t),\theta_2(t))}{\partial\theta_2(t)}=\varphi(\theta_1(t),I_1)$. Both these differentiates are continuous in $\theta_1(t)$ and $\theta_2(t)$. Using the Implicit function theorem, we know that in cases where $\frac{\partial\mathcal{G}(\theta_1(t),\theta_2(t))}{\partial\theta_2(t)}\neq 0 \Leftrightarrow \theta_1(t)\neq i_1$, there exists a unique function $\theta_2(t)=\psi(\theta_1(t))$, defined and continuous in a neighborhood of $\theta_1(t)\in\mathcal{C}\cap\{\theta_1(t)\neq i_1\}$ such that $\mathcal{G}\Big(\theta_1(t),\psi(\theta_1(t))\Big)=0$.

By substitution of $\psi(.)$ in Equation (8) and using the fact that $\mathcal{G}\Big(\theta_1(t),\psi(\theta_1(t))\Big)=0$, we obtain the analytic expression of function $\psi(.)$ which enables us to express $\theta_2(t)$ as a function of $\theta_1(t)$ solely, i.e.:

$$\begin{aligned}\theta_2(t) &= \psi\Big(\theta_1(t)\Big)\\ &= \Big(1-\theta_1(t)\Big)+\frac{p_1(t)+\bar{d}_1(t)-c_{\max}}{\theta_1(t)-i_1}.\end{aligned}$$

System $\mathfrak{G}(2)$ can then be rewritten by expliciting $\theta_2(t)$ as a function of $\theta_1(t)$ exclusively. It gives us a new system of equations that we call $\mathfrak{G}(3)$.

$$\mathfrak{G}(3)\begin{cases} \theta_2(t)=\psi\Big(\theta_1(t)\Big) \\ \qquad\quad =(1-\theta_1(t))+\frac{p_1(t)+\bar{d}_1(t)-c_{\max}}{\theta_1(t)-\kappa_1} \\ (\theta_1(t),\theta_2(t))\in\mathcal{C} \end{cases}$$

The solutions of System $\mathfrak{G}(3)$ (provided they exist) are contained in the solutions of System $\mathfrak{G}(1)$. Furthermore, if we impose the additional constraint that $\theta_1(t)+\theta_2(t)=x$ for any $0<x<1$ in System $\mathfrak{G}(3)$, then making x vary enables to browse all the solutions of System $\mathfrak{G}(1)$. □

3.2 Optimal pricing

In this section, we solve step (2) of the repeated Stackelberg game described in Section 2. As usual in multi-level games, we proceed by backward induction [3], [15]. Step (3) has already been solved in Section 3.1: the EV drivers allocation between the service providers are obtained through System $\mathfrak{S}(3)$ which relies on service provider S_1's price $p_1(t)$. Going to step (2), we want to determine the prices $p_1(t), p_2(t)$ maximizing the providers' utilities as defined in Equation (1) and taking into account the price constraint defined in Equation (5).

The price ordering obtained in Equation (5) can be decomposed into the following constraints, giving intervals of definition for both service providers' prices:

- $p_E(t) \leq p_1(t) < p_2(t) - \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big)$ for provider S_1,
- $p_1(t) + \Big(\bar{d}_1(t) - \bar{d}_2(t)\Big) < p_2(t) \leq p_{\max}$ for provider S_2.

The utilities defined in Equation (1) being continuous, the service providers' optimal prices are reached either in their interior or at one boundary of their price interval of definition. Besides, System $\mathfrak{S}(3)$ depends only on service provider S_1's price which in turn, affects provider S_2's pricing strategy since the upper bound of provider S_2's price interval of definition depends on $p_1(t)$. Depending on whether the optimal prices are reached at the interior or at one boundary of the interval of definition, 4 cases emerge. They are detailed below.

3.2.1 Case 1: The service providers' optimal prices are reached in the interior of their price intervals of definition

It means that $p_1(t) \in]p_E(t); p_2(t) - (\bar{d}_1(t) - \bar{d}_2(t))[$ and $p_2(t) \in]p_1(t) + (\bar{d}_1(t) - \bar{d}_2(t)); p_{\max}[$. At the equilibrium, for any $k = 1, 2$, provider S_k's utility should be solution of the following equation: $\frac{\partial \pi_k(t)}{\partial p_k(t)} = 0$. Then, for any provider S_k, $k = 1, 2$, we have the following equivalences:

$$
\begin{aligned}
\frac{\partial \pi_k(t)}{\partial p_k(t)} = 0 \quad &\Leftrightarrow \quad \theta_k(t) + (p_k(t) - p_E(t))\frac{\partial \theta_k(t)}{\partial p_k(t)} = 0 \\
&\Leftrightarrow \quad \frac{\partial \theta_k(t)}{\theta_k(t)} = -\frac{\partial p_k(t)}{p_k(t) - p_E(t)} \\
&\Leftrightarrow \quad \log(\theta_k(t)) = \log(\frac{1}{p_k(t) - p_E(t)}) \\
&\Leftrightarrow \quad \theta_k(t) = \frac{1}{p_k(t) - p_E(t)} \text{ since log is a bijective function.}
\end{aligned}
$$

We infer that $p_k(t) = p_E(t) + \frac{1}{\theta_k(t)}$ for any provider $S_k,\ k = 1, 2$ and by substitution in System $\mathfrak{S}(3)$, we obtain a new system of equations:

$$\mathfrak{G}(3,1)\begin{cases}\theta_2(t)=\psi(\theta_1(t))\\ \qquad\quad=\Big(1-\theta_1(t)\Big)+\frac{1}{\theta_1(t)-i_1}\Big[p_E(t)+\frac{1}{\theta_1(t)}+\bar{d}_1(t)-c_{\max}\Big]\\ (\theta_1(t),\theta_2(t))\in\mathcal{C}\end{cases}$$

The objective is now to prove the existence and unicity of solutions for System $\mathfrak{G}(3,1)$.

Unicity of solution for System $\mathfrak{G}(3,1)$. In the following lemma, we prove the unicity of System $\mathfrak{G}(3,1)$ solution for any value of the total market share[4] $x\in[0;1]$ such that $\theta_1(t)+\theta_2(t)=x$.

Lemma 5. *For any $x\in[0;1]$ there exists a unique couple of value $(\theta_1(t),\theta_2(t))$ such that $\theta_1(t)+\theta_2(t)=x$ and $\theta_2(t)=\psi(\theta_1(t))$.*

Proof of Lemma 5. It can be found in Appendix.

Although we have proved that $\psi(.)$ is increasing over interval $[0;1]$, we have no guarantees about its concavity yet. Indeed it might admit an inflection point over the interval. In the following lemma, we introduce a sufficient condition on $c_{\max}$, the EV drivers' maximum admissible opportunity cost, guaranteeing the concavity of function $\psi(.)$ over interval $[0;1]$.

Lemma 6. *If $c_{\max}\leq(1-i_1)+p_E(t)+\bar{d}_1(t)$ then $\psi(.)$ is concave in $\theta_1(t)$.*

Proof of Lemma 6. It can be found in Appendix.

Existence of solution for System $\mathfrak{G}(3,1)$. In the following proposition, we detail conditions on i_1,i_2 guaranteeing the existence of solutions for System $\mathfrak{G}(3,1)$, for any fixed value of the total market share $x\in[0;1]$ such that $\theta_1(t)+\theta_2(t)=x$.

We introduce two three-order polynoms that will be useful in the following proposition proof:

$$\begin{aligned}P_1\Big(\theta_1(t)\Big)\quad&\equiv\quad 2\theta_1^3(t)-\theta_1^2(t)\Big[1+2i_1+\Big(\tilde{\varphi}(I_1)-\tilde{\varphi}(I_2)\Big)\Big]-\theta_1(t)\Big[-i_1\Big(1\\ &+\quad(\tilde{\varphi}(I_1)-\tilde{\varphi}(I_2))\Big)+p_E(t)+\bar{d}_1(t)-c_{\max}\Big]-1\end{aligned}$$

and

$$P_2\Big(\theta_1(t)\Big)\equiv\theta_1^3(t)-\theta_1^2(t)(1+i_1)+\theta_1(t)\Big(\kappa_1-p_E(t)-\bar{d}_1(t)+c_{\max}\Big)-1$$

The limits in $-\infty$ and $+\infty$ of $P_1(.)$ (resp. $P_2(.)$) being infinite and of opposite signs, function $P_1(.)$ (resp. $P_2(.)$) being continuous since polynomial, the intermediate value theorem provides us the guarantee of the existence of at least a real root solution of the equation: $P_1(.)=0$ (resp. $P_2(.)=0$). The three roots associated with polynom $P_1(.)$ are denoted $r_{P_1}(l),\ l=1,2,3$ (resp. $r_{P_2}(l),\ l=1,2,3$ for polynom $P_2(.)$).

[4]In economics, the total market share i.e., the sum of the rival providers' market share is also called penetration or market coverage rate [13].

We give necessary and sufficient conditions on i_1, i_2 guaranteeing that System $\mathfrak{G}(3,1)$ admits a solution.

Proposition 7. *Suppose providers S_1 and S_2's optimal prices are reached in the interior of their price intervals of definition.*

- *In case where $\theta_1(t) < i_1$ (no congestion), System $\mathfrak{G}(3,1)$ admits a solution if, and only if, i_1, i_2 are chosen so that*

$$\begin{aligned}
&i_1 \leq \frac{1}{c_{\max} - \bar{d}_1(t) - p_E(t)} \\
&\theta_1(t) \in [0; r_{P_1}(1)[\cup]r_{P_1}(1); r_{P_1}(3)[\\
&\theta_1(t) \in [r_{P_2}(1); r_{P_2}(2)] \ \cup \ [r_{P_2}(3); +\infty[\\
&\theta_1(t) \in [0; 1]
\end{aligned}$$

- *In case where $\theta_1(t) \geq i_1$ (congestion occurs), System $\mathfrak{G}(3,1)$ admits a solution if, and only if, i_1, i_2 are chosen so that*

$$\begin{aligned}
&i_1 > \frac{1}{c_{\max} - p_E(t) - \bar{d}_1(t)} \\
&\theta_1(t) \in [r_{P_1}(1); r_{P_1}(2)] \ \cup \ [r_{P_1}(3); +\infty[\\
&\theta_1(t) \in [0; r_{P_2}(1)[\cup]r_{P_2}(2); r_{P_2}(3)[\\
&\theta_1(t) \in [0; 1]
\end{aligned}$$

Proof of Proposition 7. It can be found in Appendix.

As a corollary of Proposition 7, if we impose that the whole market is captured i.e., $\theta_1(t) + \theta_2(t) = 1$ then the EV driver allocation is unique: $\theta_1(t) = \frac{1}{c_{\max} - p_E(t) - \bar{d}_1(t)}$ and $\theta_2(t) = 1 - \theta_1(t)$. In all generalities, if we fix the number of EV drivers delaying their reload as $1 - x \in]0; 1]$ then the optimal number of consumers for provider S_1 satisfying $\theta_1(t) + \theta_2(t) = x$ is defined uniquely as:

$$\begin{aligned}
\theta_1(t) \quad &= \quad \frac{1}{2}\Big(\frac{1}{x-1}(p_E(t) + \bar{d}_1(t) - c_{\max}) + i_1 \\
&- \quad \sqrt{(\frac{1}{1-x}(p_E(t) + \bar{d}_1(t) - c_{\max}) + i_1)^2 + \frac{4}{1 - p_E(t)}}\Big).
\end{aligned}$$

Then $\theta_2(t)$ is inferred from the relation: $\theta_2(t) = x - \theta_1(t)$.

3.2.2 Case 2: Provider S_1's optimal price is reached at the inferior bound of his price interval of definition.

In this case, $p_1(t) = p_E(t)$ and $p_E(t) + (\bar{d}_1(t) - \bar{d}_2(t)) < p_2(t) \leq p_{\max}$. By substitution of provider S_1's optimal price in System $\mathfrak{G}(3)$, we obtain:

$$\mathfrak{G}(3,2) \begin{cases} \theta_2(t) = \psi(\theta_1(t)) \\ \qquad\quad = (1 - \theta_1(t)) + \frac{1}{\theta_1(t) - i_1}\Big[p_E(t) + \bar{d}_1(t) - c_{\max}\Big] \\ (\theta_1(t), \theta_2(t)) \in \mathcal{C} \end{cases} \tag{9}$$

Proposition 8. *Suppose provider S_1's optimal price is reached at the inferior bound of his price interval of definition. System $\mathfrak{S}(3,2)$ admits a unique solution $\theta_1(t) = 0$ and $\theta_2(t) = 1 - \frac{1}{i_1}[p_E(t) + \bar{d}_1(t) - c_{\max}]$ if, and only if, $c_{\max} \leq p_E(t) + \bar{d}_1(t)$.*

Proof of Proposition 8. It can be found in Appendix.

As a corollary of Proposition 8, the whole market is captured by provider S_2 if, and only if, $c_{\max} = p_E(t) + \bar{d}_1(t)$. In all the other cases, it is impossible for the provider to capture the total market.

3.2.3 Case 3: Provider S_1's optimal price is reached at the upper bound of his price interval of definition and provider S_2 in the interior of his price interval of definition

Under these assumptions, there exists a real $\varepsilon > 0$ such that $\varepsilon \to 0$ and $p_1(t) = p_2(t) - (\bar{d}_1(t) - \bar{d}_2(t)) - \varepsilon$. Provider S_2 price constraint becomes: $p_1(t) + (\bar{d}_1(t) - \bar{d}_2(t)) < p_2(t) \leq p_{\max}$. Since S_2 reaches his optimal price in the interior of the interval, we have: $p_2(t) = p_E(t) + \frac{1}{\theta_2(t)}$.

Proposition 9. *Suppose provider S_1's optimal price is reached at the upper bound of his price interval of definition and provider S_2 in the interior of his price interval of definition. The relations between the providers' market shares take the following forms depending whether congestion occurs.*

- *If $\theta_1(t) > i_1$ (congestion occurs),*

$$\theta_2(t) = \frac{1}{2}\Big[(1 - \theta_1(t)) + \frac{1}{\theta_1(t) - i_1}(p_E(t) + \bar{d}_1(t) - c_{\max}) + \sqrt{[(\theta_1(t) - 1) - \frac{1}{\theta_1(t) - i_1}(p_E(t) + \bar{d}_1(t) - c_{\max})]^2 + \frac{4}{\theta_1(t) - i_1}}\Big].$$

- *If $\theta_1(t) \leq i_1$ (no congestion),*

$$\theta_2(t) = \frac{1}{2}\Big[(1 - \theta_1(t)) + \frac{1}{\theta_1(t) - i_1}(p_E(t) + \bar{d}_1(t) - c_{\max}) - \sqrt{[(\theta_1(t) - 1) - \frac{1}{\theta_1(t) - i_1}(p_E(t) + \bar{d}_1(t) - c_{\max})]^2 + \frac{4}{\theta_1(t) - i_1}}\Big].$$

Proof of Proposition 9. It can be found in Appendix.

As a corollary of Proposition 9, if we assume that the whole market is captured we infer each provider's market share: $\theta_2(t) = \frac{1}{(\varepsilon + c_{\max}) - (\bar{d}_2(t) + p_E(t))}$ and $\theta_1(t) = 1 - \theta_2(t)$.

3.2.4 Case 4: Both providers reach their optimal prices at the upper bounds of their own price interval of definition

It means that $p_2(t) = p_{\max}$ from which we infer $p_1(t) = p_{\max} - (\bar{d}_1(t) - \bar{d}_2(t)) - \varepsilon$.

Proposition 10. *Suppose that both providers reach their optimal prices at the upper bounds of their own price interval of definition, the relation between the providers' market shares takes the following form:*

$$\theta_2(t) = (1 - \theta_1(t)) + \frac{p_{\max} - \varepsilon + (\bar{d}_2(t) - c_{\max})}{\theta_1(t) - i_1}$$

Proof of Proposition 10. It is straightforward by substitution of provider S_1's optimal price in the equation: $\theta_2(t) = \psi(\theta_1(t))$ described in System $\mathfrak{G}(3)$.

□

As a corollary of Proposition 10, if we assume that the whole market is captured, we infer that: $c_{\max} = p_{\max} - \varepsilon + \bar{d}_2(t)$. This means that no EV drivers would choose provider S_2. Therefore, $\theta_1(t) = 1$ and $\theta_2(t) = 0$.

4 Optimization of the charging infrastructure topologies using the Cross-Entropy method

In this section, the service providers determine simultaneously and independently the number of stations that they wish to settle as well as their locations on the $\mathbb{R}^2$ plane. This coincides with step (1) of the game described in Section 2. Step (1) is solved so as to optimize the service provider's utility over the complete duration of the game i.e., over time interval $[0; T]$. For this purpose, we need to define service provider $S_k,\ k = 1, 2$'s *long-term utility*. Indeed, in general, the optimal decisions of a player differ in cases where we consider a repeated game that leads to a sequencial optimization problem and in cases where the decisions are optimized over the entire course of the game. In our game setting, provider $S_k,\ k = 1, 2$ chooses $N_k \in \mathbb{N}^*$ as the number of stations to settle and $s_k \in \mathbb{R}^{2N_k},\ k = 1, 2$ as their locations on the $\mathbb{R}^2$ plane. His long-term utility is the sum of his one-shot utilities as described in Equation (1) which outputs are added sequentially over the discretized finite horizon $[0; T]$, coefficiented by a discount factor $\delta \in]0; 1]$[5]:

$$\Pi_k(s_1, s_2) = \sum_{t=0}^{T} \delta^t \pi_k(t) \tag{10}$$

$x \in [0; 1]$ is the total market share i.e., the sum of both providers' market shares. It is known publicly by all the providers and satisfies: $\theta_1(t) + \theta_2(t) = x$ at any time

[5]The discount factor is supposed identical for both providers. It means that both of them has the same risk aversion level for the future [28], [34]. The introduction of a discount factor is classical in (stochastic) dynamic programming and repeated game theory where the players consider their long-term utilities and have uncertainties on the future. However, extensions where it differs between the rival providers could be considered in a companion paper more devoted to simulation results.

instant t. It is supposed constant over time since the batteries have a limited autonomy which force the EV drivers to reload periodically in time. Using the total market share value, Cases 1, 2 and 4 as described in Section 3.2, can be solved analytically. In these cases, the resolution of the pricing game requires to solve polynomial equations in the market shares. Case 3 is more complex and requires a numerical resolution to identify the service providers' market shares. However, in all the cases, it is impossible to determine analytically the service providers' optimal number of stations as well as their locations, since the EV drivers' choices depend on their spatial distribution on the plane which is modeled as a time-dependent density function in Equation (3). Consequentely, we resort to use simulation to determine the optimal topologies of the service providers' charging infrastructures.

We decouple the problem of determining the optimal number of stations to settle in the $\mathbb{R}^2$ plane which is solved numerically from the problem of optimizing the station locations on the $\mathbb{R}^2$ plane which is solved through simulation. We now describe both phases. For any service provider S_k, $k = 1, 2$, the problem of optimizing his charging infrastructure topology can be written as:

$$\gamma_k^* = \Pi_k(s_1^*, s_2^*) = \max_{s_k \in \mathbb{R}^{2N_k}} \Pi_k(s_k, s_{\{1,2\}-k}^*) \tag{11}$$

where $s_{\{1,2\}-k}^*$ denotes the vector containing the station locations of the provider different from S_k for $k = 1, 2$, γ_k^* is the maximum long-term utility output for service provider S_k. It is not necessarily unique and might be different for the rival providers.

To determine the charging infrastructure topologies optimizing Equation (11), we transform the initial deterministic optimization problem in an *estimation one*. The cross-entropy method introduced by Rubinstein et al. [8], [27] is a general Monte-Carlo approach to combinatorial problems like the travelling salesman, quadratic assignment, etc., continuous multi-extremal optimization and importance sampling [11]. It originates from the field of rare event simulation where very small probabilities need to be accurately estimated like in queueing theory applications or more generally, performance analysis of complex systems. The method has been successfully applied to problems belonging to the field of combinatorial optimization [12] which are considered to be hard to solve. There have been many successful applications of the method to diverse fields such as routing and performance evaluation in telecommunication networks [27], queueing theory, sensor selection/management [31], the optimization of the locations of a sparse antenna array [20], etc. Le Cadre et al. use it to solve the global level when decomposing a hierarchical search problem in two optimization levels. They check that it enables to find optimal solutions in most cases with a reasonable time [31]. These points motivated us in our turn to use the method. Additionally, we obtain as output of the algorithm the optimal density function generating the charge station locations which is quite original compared to previous approaches in combinatorial optimization [5] and might facilitate potential computations of performance measures.

To apply the cross-entropy method to our problem, we need to suppose that provider $S_k,\ k = 1, 2$ station locations are distributed according to a density parametrized by a

multi-dimensional vector of parameters called $\rho_k,\ k = 1, 2$ i.e.,
$s_k = (s_k(1), ..., s_k(N_k)) \sim f(., \rho_k)$ where $\rho_k \in \mathbb{R}^{2N_k}$. We let γ_k be a fixed level for service provider S_k's long-term utility. For a certain $\rho_k \in \mathbb{R}^{2N_k}$, we associate with (11), the problem of estimating the number $l(\gamma_k)$:

$$\begin{aligned} l(\gamma_k) &= \mathbb{P}_{\rho_k}[\Pi_k(s, s^*_{\{1,2\}-k}) \geq \gamma_k] = \int_{s \in \mathbb{R}^{2N_k}} \mathbf{1}_{\{\Pi_k(s, s^*_{\{1,2\}-k}) \geq \gamma_k\}} f(s, \rho_k) ds \\ &= \mathbb{E}_{\rho_k}[\mathbf{1}_{\{\Pi_k(s, s^*_{\{1,2\}-k}) \geq \gamma_k\}}] \end{aligned} \tag{12}$$

We give below a description of the algorithm that we use to optimize the service providers' charging infrastructure topologies. It is based on updates of density parameters $\rho_k \in \mathbb{R}^{2N_k},\ k = 1, 2$.

4.1 Algorithm description

We take service provider $S_k,\ k = 1, 2$'s point of view. The algorithm is based on the cross-entropy method introduced by Rubinstein et al. in [8], [27]. We use the following convention: for any real number $r \in \mathbb{R}$, $\lceil r \rceil$ corresponds to the smallest integer number superior to r.

Algorithm: Charging infrastructure topology optimization

Parameters:

- $M \in \mathbb{N}^*$ number of iterations
- $\mathcal{N}_s \in \mathbb{N}^*$ sample size
- $\zeta \in]0; 1[$ one minus the value of the quantile associated to the performance statistics
- $d \in \mathbb{N}^*$ stopping criterion

The algorithm is run for $m = 1, ..., M$ iterations.

1. Define $\hat{\rho}_k(0)$. Set $m = 1$.
2. For any $m \in \{1, ..., M\}$. Generate a sample $(s_k^1, ..., s_k^{\mathcal{N}_s}) \sim f(., \hat{\rho}_k(m-1))$ where $s_k^l \in \mathbb{R}^{2N_k},\ \forall l = 1, ..., \mathcal{N}_s$. Let $\Pi_k^1 = \Pi_k(s_k^1, s^*_{\{1,2\}-k})$,..., $\Pi_k^{\mathcal{N}_s} = \Pi_k(s_k^{\mathcal{N}_s}, s^*_{\{1,2\}-k})$ and determine the associated ordered statistic: $\Pi_k^{(1)} \leq ... \leq \Pi_k^{(\mathcal{N}_s)}$. Compute the $(1-\zeta)$ quantile $\hat{\gamma}_k(m)$ of the performance according to $\hat{\gamma}_k(m) = \Pi_k(\lceil (1-\zeta)\mathcal{N}_s \rceil)$.
3. Use the same sample $s_k^1, ..., s_k^{\mathcal{N}_s}$ and solve:

$$\hat{\rho}_k(m) = \max_{\rho_k \in \mathbb{R}^{2N_k}} \frac{1}{\mathcal{N}_s} \sum_{l=1}^{\mathcal{N}_s} \mathbf{1}_{\Pi_k(s_k^l, s^*_{\{1,2\}-k}) \geq \hat{\gamma}_k(m)} \log f(s_k^l, \rho_k) \tag{13}$$

4. If for some $m \geq d$ (ex.: $d = 5$), $\hat{\rho}_k(m) = \hat{\rho}_k(m-1) = ... = \hat{\rho}_k(m-d)$ then stop. Otherwise set $m = m + 1$ and go back to step 2 of the algorithm.

Making some assumptions on the station location generating density, we now detail how to solve Equation (13).

For the sake of simplicity, service provider S_k's l-th station coordinates are stored in the two-dimensional coordinate vector:

$$s_k(l) = (s_k(l)|_x, s_k(l)|_y) \in \mathbb{R}^2$$

where $s_k(l)|_x$ contains vector $s_k(l)$ projection on the x-axis and $s_k(l)|_y$, on the y-axis.

We assume that there exists a bijective mapping between the station coordinates and the first $2N_k$ integers:

$$\mathfrak{M} : s_k(1)|_x, s_k(1)|_y, ..., s_k(N_k)|_x, s_k(N_k)|_y \mapsto \{1, 2, ..., 2N_k - 1, 2N_k\}$$

Practically, this means that two charge stations settled by provider S_k cannot share identical x-axis (resp. y-axis) coordinates. Using the same idea, for any generated sample $l = 1, ..., \mathcal{N}_s$, there exists a bijective application $\mathfrak{M}^l(.)$, mapping the l-th coordinate vector s_k^l on the first $2N_k$ integers:

$$\mathfrak{M}^l : s_k^l(1)|_x, s_k^l(1)|_y, ..., s_k^l(N_k)|_x, s_k^l(N_k)|_y \mapsto \{1, 2, ..., 2N_k - 1, 2N_k\}$$

Besides, we assume that at iteration step $m \in \{1, ..., M\}$ of the algorithm, service provider S_k's station locations are distributed according to a $2N_k$- dimensional normal density with covariance matrix:
$\sigma_k = \begin{pmatrix} \sigma_k(1,1) & \dots & \sigma_k(1, 2N_k) \\ \vdots & \vdots & \vdots \\ \sigma_k(2N_k, 1) & \dots & \sigma_k(2N_k, 2N_k) \end{pmatrix}$. The covariance matrix contains the uncertainty levels associated with the knowledge of the service providers' station locations. It is supposed fixed a priori and known publicly. We make the assumption that the covariance matrix coefficients remaining on its diagonal always remain positive. They coincide with the variances associated with provider S_k's stations coordinates.

The mean $\hat{\rho}_k(m) \in \mathbb{R}^{2N_k}$ associated with the $2N_k$-components of the normal density are unknown and should be estimated through Equation (13) solving. Differentiating Equation (13) with respect to the mean components, we obtain that it is equivalent to solve the following matricial system:

$$\hat{\rho}_k(m) = \frac{1}{\sum_{l=1}^{N} \mathbf{1}_{\Pi_k(s_k^l, s^*_{\{i,j\}-k}) \geq \hat{\gamma}_k(m)}} \sum_{l=1}^{N} \mathbf{1}_{\Pi_k(s_k^l, s^*_{\{i,j\}-k}) \geq \hat{\gamma}_k(m)} R_k^{-1} C_k^l$$

where $C_k^l = \begin{pmatrix} s_k^l(1)|x + \frac{1}{2}\sum_{h\neq 1} \frac{\sigma_k(1,h)}{\sigma_k(1,1)}\mathfrak{M}^{l,-1}(h) \\ s_k^l(1)|y + \frac{1}{2}\sum_{h\neq 2} \frac{\sigma_k(2,h)}{\sigma_k(2,2)}\mathfrak{M}^{l,-1}(h) \\ \vdots \\ s_k^l(N_k)|x + \frac{1}{2}\sum_{h\neq 2N_k-1} \frac{\sigma_k(2N_k-1,h)}{\sigma_k(2N_k-1,2N_k-1)}\mathfrak{M}^{l,-1}(h) \\ s_k^l(N_k)|y + \frac{1}{2}\sum_{h\neq 2N_k} \frac{\sigma_k(2N_k,h)}{\sigma_k(2N_k,2N_k)}\mathfrak{M}^{l,-1}(h) \end{pmatrix}$ and $\mathfrak{M}^{l,-1}(.)$ denotes the inverse of mapping $\mathfrak{M}^l(.)$ associated with the l-th generated sequence.

R_k is a $2N_k \times 2N_k$ matrix defined so that $R_k(i,i) = 1,\ \forall i = 1, ..., 2N_k$ and $R_k(j,i) = \frac{1}{2}\frac{\sigma_k(j,i)}{\sigma_k(j,j)} \geq 0,\ \forall i,j = 1,...,2N_k,\ \ i \neq j$. Note that using the covariance definition, R_k is semi-positive definite and we observe that: $R_k(j,i) = R_k(i,j),\ \forall i,j = 1,...,2N_k,\ \ i \neq j$. This last point implies that matrix R_k is symmetric.

Lemma 11. *Matrix R_k is invertible.*

Proof of Lemma 11. It is trivial to check that R_k is invertible. Using the covariance definition, we have the inequality:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \leq R_k$$

where inequality between two matrices of the same size is defined coordinate per coordinate. Determinant being a linear form, we infer that: $1 \leq \det(R_k)$. $\square$

The results obtained above are summarized in the following theorem.

Theorem 12. *Assuming that service provider S_k's stations locations are distributed according to a $2N_k$- dimensional normal density centered in vector $\hat{\rho}_k(m)$ and of fixed covariance matrix σ_k, the updating rule of vector $\hat{\rho}_k(m)$ as described in Equation (13) is equivalent with the matricial equation:*

$$\hat{\rho}_k(m) = \frac{1}{\sum_{l=1}^{\mathcal{N}_s} \mathbf{1}_{\Pi_k(s_k^l, s^*_{\{1,2\}-k}) \geq \hat{\gamma}_k(m)}} \sum_{l=1}^{\mathcal{N}_s} \mathbf{1}_{\Pi_k(s_k^l, s^*_{\{1,2\}-k}) \geq \hat{\gamma}_k(m)} R_k^{-1} C_k^l$$

where $C_k^l = \begin{pmatrix} s_k^l(1)|x + \frac{1}{2}\sum_{j\neq 1} \frac{\sigma_k(1,j)}{\sigma_k(1,1)}\mathfrak{M}^{l,-1}(j) \\ s_k^l(1)|y + \frac{1}{2}\sum_{j\neq 2} \frac{\sigma_k(2,j)}{\sigma_k(2,2)}\mathfrak{M}^{l,-1}(j) \\ \vdots \\ s_k^l(N_k)|x + \frac{1}{2}\sum_{j\neq 2N_k-1} \frac{\sigma_k(2N_k-1,h)}{\sigma_k(2N_k-1,2N_k-1)}\mathfrak{M}^{l,-1}(j) \\ s_k^l(N_k)|y + \frac{1}{2}\sum_{j\neq 2N_k} \frac{\sigma_k(2N_k,j)}{\sigma_k(2N_k,2N_k)}\mathfrak{M}^{l,-1}(j) \end{pmatrix}$ *and R_k is a symmetric invertible $2N_k \times 2N_k$ matrix defined so that $R_k(j,j) = 1,\ \forall j = 1, ..., 2N_k$ and $R_k(i,j) = \frac{1}{2}\frac{\sigma_k(i,j)}{\sigma_k(i,i)} \geq 0,\ \forall i,j = 1,...,2N_k,\ i \neq j$.*

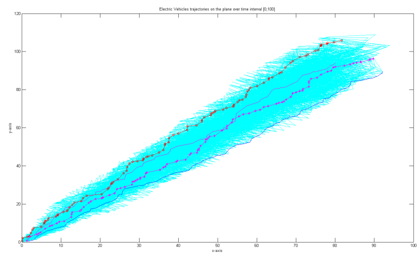

Figure 1: Cone of the simulated EV trajectories.

5 Numerical illustrations

The aim of this section is to explain practical realisations of the model elaborated in this article and to provide some economic guidelines.

The total number of EV on the market is supposed constant: $N = 100$. However, the number of EVs which reload at each time instant t is defined as a fraction of the total number of EVs i.e., there exists a random variable $\alpha(t) \in [0; 1]$ such that $N(t) = \alpha(t)N$. In the numerical analysis, we assume that $\alpha(t)$ is generated according to the uniform density over interval $[0; 1]$. Additionally, the EV drivers' trajectories are generated according to independent two-dimensional brownian motions whose components are correlated. N independent brownian motions are generated. To generate N correlated brownian motions, we need to specify the correlation matrix between the components of these brownian motions: Q, which is of size $N \times N$. At time instant t, the matrix containing the variances-covariances of the N correlated Brownian motion equals: $Q(t) = tQ$. Then, the Choleski decomposition of matrix $Q(t)$ enables us to express the correlated brownian motion components as a linear combination of the N independent brownian motions.

The mean value of the distance between the EV drivers and provider $S_k,\ k = 1, 2$ is obtained as an approximation of Equation (3). At each time instant t, once the EV locations on the $\mathbb{R}^2$ plane has been obtained through the realizations of the $N(t)$ correlated brownians motions, for each provider S_k, we compute the minimum of their distance to the closest station of the provider and average this value over $N(t)$.

As an illustration, in Figure 1, we have simulated the trajectories of 4 EVs starting from the same origin $(0; 0)$, over time interval $[0; 100]$. The cone of the 100 simulated trajectories for the EVs is represented in light blue.

In all this section, the game parameters are fixed as follows:

- The discount factor is: $\delta = 0.7$.
- The service providers' capacity per station are: $\mu_1 = 0.83, \mu_2 = 0.90$.

- The providers' investment level increases linearly in the number of stations settled i.e.: $I_1 = 0.01N_1 + 0.05$ and $I_2 = 0.03N_2 + 0.02$.
- Provider $S_k,\ k = 1, 2$'s extended capacity is defined as: $i_k = \mu_k + 0.1I_k$.
- The total market share also called market coverage in the article, is: $x = 0.8$.
- The repeated pricing game parameters are defined as: $T = 100$, $p_{\max} = 50$, $c_{\max} = 100$, $p_E = 10$, $c_E = 10$, $\varepsilon = 10^{-3}$.
- σ_1 and σ_2 are non-negative symmetric matrices whose upper-diagonal and diagonal coefficients are generated according to a uniform density on interval $[10^{-3}; 1]$.
- For the cross-entropy algorithm, the sample size is fixed so that: $\mathcal{N}_s = 10$ and the associated quantile is: $\zeta = 0.7$. Finally, the cross-entropy algorithm maximum number of iterations is: $M = 100$.

5.1 Optimization of the charging infrastructure topologies

Each provider can settle: 5, 15 or 25 stations over the $\mathbb{R}^2$ plane. This gives rise to 3^2 potential combinations. The long-term utilities of both providers are stored in Table 1 below, for each potential combination of numbers of stations.

N_1 \ N_2	5	15	25
5	$(-23.38; 204.97)$	$(-97.55; 333.67)$	$(\mathbf{443.93}; \mathbf{962.40})$
15	$(649.97; -23.04)$	$(-57.65; 691.24)$	$(510.34; 601.63)$
25	$(\mathbf{895.43}; \mathbf{231.71})$	$(64.62; -319.58)$	$(-21.55; -531.99)$

Table 1: Providers' long-term utilities as functions of the number of settled stations.

We observe that there are two pure Nash equilibria (NE) in Table 1. They are both highlighted in bold and correspond to the cases: $N_1 = 5, N_2 = 25$ for the first NE and $N_1 = 25, N_2 = 5$ for the second one. NE $N_1 = 5, N_2 = 25$ is more favorable to provider S_2 who maximizes his long-term utility over all the potential combinations of number of stations whereas NE $N_1 = 25, N_2 = 5$ is more favorable to provider S_1. As a result, the system might become instable if the providers does not agree and generate periodic go-backs between the two NEs. However, if it is an unbiased decision maker who drives the system, he will force the system to stabilize in the first NE. Indeed, this latter coincides with the game maximum social welfare i.e., it maximizes the sum of both providers' long-term utilities.

In Figure 2, we have represented the providers' long-term utilities obtained as output of the dynamic pricing game, as functions of the number of stations settled by each provider. The two NEs are highlighted by squares.

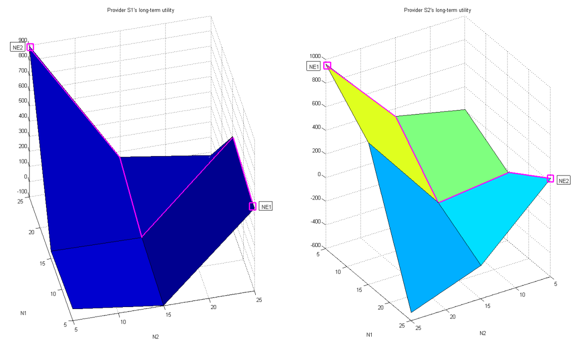

Figure 2: Numerical determination of the number of stations to settle.

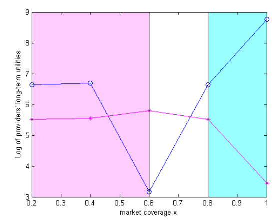

Figure 3: Providers' long-term utilities as functions of their market share.

5.2 Market coverage

In Figure 3, we have represented the logarithm of both providers' long-term utilities (provider S_1 in blue with o and provider S_2 in magenta with $*$) as functions of the market coverage called $x \in [0; 1]$ in all the article. We observe that provider S_1's long-term utility is increasing provided $0 \leq x \leq 0.6$ whereas provider S_2's long-term utility is increasing provided $0.8 \leq x \leq 1$. This enables us to define intervals for x, the first one being favorable to S_2 and the second one to S_1. The value $x = 0.8$ appears as good compromise since it is not penalizing any of the providers.

5.3 An alliance to share the investments in the charging infrastructure?

Sorensen indexes are traditionally used in ecology to characterize the similarity in terms of species present on two geographic area [2]. We make the analogy with our illustration by defining 4 species: none of the providers (spece 0); provider S_1 exclusively (spece 1); provider S_2 exclusively (spece 2); both providers (spece 3). To define geographic areas, we realize a mesh over the $\mathbb{R}^2$ plane. It is delimitated by the extreme coordinates of the settled stations:

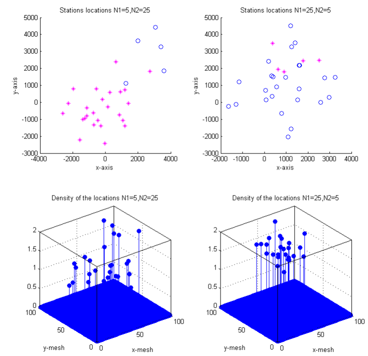

Figure 4: Density of the station locations.

$\Big[\min_{k=1,2}\{\min_{l=1,\dots,N_k} s_k(l)|_x\}; \max_{k=1,2}\{\max_{l=1,\dots,N_k} s_k(l)|_x\}\Big] \times \Big[\min_{k=1,2}\{\min_{l=1,\dots,N_k} s_k(l)|_y\}; \max_{k=1,2}\{\max_{l=1,\dots,N_k} s_k(l)|_y\}\Big]$. It is then divided into a collection of squares of equal size. Over each axis, the division step is fixed at 100 giving rise to 100^2 squares of equal size.

In top of Figure 4, we have plotted the optimal stations locations over the $\mathbb{R}^2$ plane for the first NE ($N_1 = 5, N_2 = 25$) at left and for the second NE ($N_1 = 25, N_2 = 5$) at right. At the bottom of the figure, we have represented the number of species ranking from 0 to 3 present in each square of the mesh area. In case of the first NE, the station locations are complementary i.e., each provider covers a closed area of the plane and the intersection between both coverage area is negligeable. In case of the second NE, the stations are more densely concentrated and there appears a conflict in the locations. The complementary configuration seems more promising since it generates higher long-term utilities than the conflicting one as observed in Section 5.1. Additionally, it might favor the emergence of geographic alliances between the providers enabling them to share their charging infrastructure investment cost while widening their coverage area. Mechanisms of cost sharing between the involved providers should then be designed so as to encourage long-term collaboration between them. This is one of the subject extensively studied in the mechanism design theory [21], [29]. It might provide possible extensions of this article.

6 Conclusion

In this article, we have considered two competitive service providers optimizing independently and simulatneously their charging infrastructure topology (in number of stations managed and in locations) while dynamically updating their station access prices so as to maximize their utility. The resulting two-stage game is solved in two steps.

First, the pricing game is solved using backward induction. We characterize analytically the unique Nash equilibrium in prices, at each time instant. Then, the chargining infrastructure is optimized using simulation to determine the station locations and numerical analysis to optimize their number.

Interesting extensions of the article might generalize the two-stage game resolution to an arbitrary large number of interacting service providers. In this case, the analytic approach provided in this article to solve the pricing game does not hold anymore. A first alternative might be to model the game as a differential one and to determine the optimal open-loop strategies in prices for the service providers [6]. A second alternative might be to consider the game as a cooperative one where coalitions might emerge [3], [21]. In such a case, the prices might be optimized using a convenient sharing mechanism for the coalitions. Another possible extension should be to add a third level to the two-stage game, by assuming that the energy provider M makes his price vary dynamically and that the energy that he can sold to the service providers at any time instant is constrained by his network capacity.

Appendix

Proof of Lemma 1

As stated above, EV driver l is indifferent between S_1 and S_2 if, and only if, $c_l(1,t) = c_l(2,t) \Leftrightarrow \beta_l = \frac{\left(p_1(t)-p_2(t)\right)+\left(\bar{d}_1(t)-\bar{d}_2(t)\right)}{\varphi\left(\theta_2(t),I_2\right)-\varphi\left(\theta_1(t),I_1\right)}$. Identically, EV driver l is indifferent between S_k and no reload if, and only if, $c_l(k,t) = c_{\max} \Leftrightarrow \beta_l = \frac{c_{\max}-p_k(t)-\bar{d}_k(t)}{\varphi\left(\theta_k(t),I_k\right)}$, for any $k = 1,2$. $\square$

Proof of Proposition 2

To determine which provider is preferred at the left or at the right of the indifference bound $B_{1,2}(t)$, it is important to determine the relative order between $q_1(t)$ and $q_2(t)$. Let consider an arbitrar EV driver l.

Suppose for instance that $q_2(t) < q_1(t)$ then $c_l(1,t) < c_l(2,t)$ i.e., provider S_1 is preferred over provider S_2 if, and only if, $\beta_l < B_{1,2}(t)$. This case is denoted Case 1.

But, if $q_1(t) < q_2(t)$ then $c_l(1,t) < c_l(2,t)$ i.e., provider S_1 is preferred over provider S_2 if, and only if, $B_{1,2}(t) < \beta_l$. This case is denoted Case 2.

Depending on the EV drivers' congestion sensitivity coefficient position on interval $[0;1]$, it is possible to determine the providers' market shares since the EV drivers' sensitivity coefficient is supposed to be distributed according to the uniform density on the interval $[0;1]$ by the assumption described in Section 2. We obtain the following market shares:

In Case 1, $\theta_2(t) = 1 - B_{1,2}(t)$, $\theta_1(t) = B_{1,2}(t) - B_{1,0}(t)$.

In Case 2, $\theta_1(t) = 1 - B_{1,2}(t)$, $\theta_2(t) = B_{1,2}(t) - B_{2,0}(t)$.

Informally, it is possible to interpret the EV drivers choices with the following arguments: When the EV drivers congestion sensitivity coefficient approaches 1, they choose the provider offering the smallest congestion level. When their sensitivity coefficient takes intermediate values, they might be less sensitive to the congestion than to the price. As a result, in this case, they accept to reload in the stations of the service provider having the highest congestion level. Finally, when their sensitivity coefficient approaches 0, the EV drivers are rather insensitive to the congestion i.e., they are not so eager to reload and can delay it. $\square$

Proof of Lemma 3

By assumption, the congestion level in S_2's charge stations is smaller that in S_1's ones since we have supposed that $q_2(t) < q_1(t)$. At time instant t, provider S_2 is always preferred over provider S_1 by any EV driver l if, and only if, $c_l(2,t) \leq c_l(1,t) \Leftrightarrow p_2(t) + \beta_l q_2(t) + \bar{d}_2(t) \leq p_1(t) + \beta_l q_1(t) + \bar{d}_1(t) \Leftrightarrow p_2(t) - p_1(t) \leq \bar{d}_1(t) - \bar{d}_2(t)$ since $\beta_l\Big(q_1(t) - q_2(t)\Big) > 0$ by assumption. $\square$

Proof of Lemma 5

Differentiating $\psi(.)$ once with respect to $\theta_1(t)$, we obtain:
$\psi'(\theta_1(t)) = -1 - \frac{1}{(\theta_1(t)-i_1)^2}(p_E(t) + \frac{1}{\theta_1(t)} + \bar{d}_1(t) - c_{\max}) - \frac{1}{\theta_1(t)-i_1}\frac{1}{\theta_1^2(t)}$. By absurd reasoning, we assume that $\psi'(\theta_1(t)) < 0$. This is equivalent with the following inequality: $-\frac{1}{(\theta_1(t)-\kappa_1)^2}(p_E(t) + \frac{1}{\theta_1(t)} + \bar{d}_1(t) - c_{\max}) + \frac{1}{\theta_1(t)-i_1}(-\frac{1}{\theta_1^2(t)}) < 1$. Multyplying each side of the inequality by the term $(\theta_1(t) - i_1)^2\theta_1(t)$, we infer: $\theta_1^4(t) - 2i_1\theta_1^3(t) + \theta_1^2(t)[i_1^2 - \bar{d}_1(t) + c_{\max} - p_E(t)] - i_1 > 0$. This inequality should be true for any $\theta_1(t) \in [0;1]$. But at the boundary $\theta_1(t) = 0$ we obtain $-i_1 > 0$ which contradicts the definition of i_1 given in Section 3.1. Therefore, $\psi'(.)$ is positive over interval $[0;1]$ meaning that function $\psi(.)$ is increasing other this interval. As a by product, the intersection of function $\psi(.)$ with the line of equation $\theta_2(t) = x - \theta_1(t)$ is unique.

$\square$

Proof of Lemma 6

Differentiating $\psi(.)$ twice with respect to $\theta_1(t)$, we obtain:

$$\begin{aligned}\psi''(\theta_1(t)) &= \frac{2(p_E(t) + \frac{1}{\theta_1(t)} + \bar{d}_1(t) - c_{\max})}{(\theta_1(t) - i_1)^3} + \frac{2}{\theta_1^3(t)(\theta_1(t) - i_1)} \\ &+ \frac{2}{\theta_1^2(t)(\theta_1(t) - i_1)^2}\end{aligned}$$

From this, we infer that $\psi(.)$ cannot be convex provided it is continuous by assumption. Reducing $\psi''(.)$ to the same denominator $\theta_1^3(t)(\theta_1(t) - i_1)^3$, we obtain $\psi''(.) < 0 \Leftrightarrow$

$\theta_1^3(t)(p_E(t)+\bar{d}_1(t)-c_{\max})+\theta_1(t)(3\theta_1(t)-2i_1)+2\kappa_1^2 < 0$. This is a polynom of order 3 in $\theta_1(t)$. The constant coefficient is positive whereas the coefficient of the higest order term $(p_E(t) + \bar{d}_1(t) + c_{\max})$ is negative. Indeed, if it were non-negative, provider S_1 would not have any clients. Besides, the polynom derivative is negative in zero meaning that the polynom is decreasing in the neighborhood of zero. As a result, if the polynom admits three real roots, there are necessarily non-negative. The polynom admits a unique minimum and a unique maximum. A sufficient condition to guaranteeing that $\theta_1^3(t)(p_E(t) + \bar{d}_1(t) - c_{\max}) + \theta_1(t)(3\theta_1(t) - 2i_1) + 2i_1^2 < 0$ is to impose that the game parameters are chosen so that the polynom minimum is greater than 1. But, the polynom minimum is reached in $\frac{6+\sqrt{36+24(p_E(t)+\bar{d}_1(t)-c_{\max})\kappa_1}}{6(c_{\max}-p_E(t)+\bar{d}_1(t))}$. Finally, this value is smaller than 1 if, and only if, $c_{\max} \leq (1 - i_1) + p_E(t) + \bar{d}_1(t)$.

□

Proof of Proposition 7

$\psi(.)$ being increasing (cf. Lemma 5), System $\mathfrak{G}(3, 1)$ admits a solution if, and only if the couple of allocations $(\theta_1(t), \theta_2(t))$ belongs to the constraint space $\mathcal{C}$. Formally, it means that the following inequalities should be simultaneously checked:

$$
\begin{aligned}
0 \leq \theta_1(t) &\leq 1 && (14)\\
\psi(\theta_1(t)) &\leq 1 - \theta_1(t) && (15)\\
\psi(\theta_1(t)) &\leq \theta_1(t) - \Big(\tilde{\varphi}(I_1) - \tilde{\varphi}(I_2)\Big) && (16)\\
\psi(\theta_1(t)) &\geq 0 && (17)
\end{aligned}
$$

Constraint (15) gives us $\frac{1}{\theta_1(t)-i_1}[p_E(t) + \frac{1}{\theta_1(t)} + \bar{d}_1(t) - c_{\max}] \leq 0$. Depending on the sign of $\theta_1(t) - i_1$, two cases appear. Either $\theta_1(t) < i_1$ i.e., there is no congestion in provider S_1's stations, or $\theta_1(t) \geq i_1$ i.e., congestion occurs. It is easy to check that if $\theta_1(t) < i_1$, the second constraint is checked if, and only if, $i_1 \leq \frac{1}{c_{\max}-\bar{d}_1(t)-p_E(t)}$. If $\theta_1(t) \geq i_1$; the second constraint is checked if, and only if, $i_1 > \frac{1}{c_{\max}-p_E(t)-\bar{d}_1(t)}$.

Constraint (16) is equivalent with $(1 - \theta_1(t)) + \frac{1}{\theta_1(t)-i_1}[p_E(t) + \frac{1}{\theta_1(t)} + \bar{d}_1(t) - c_{\max}] \leq \theta_1(t) - \Big(\tilde{\varphi}(I_1) - \tilde{\varphi}(I_2)\Big)$. To simplify the expression, we need to multiply it by $(\theta_1(t) - i_1)\theta_1(t)$. The two cases cited above already hold.

Besides, we note that $P_1(0) = -1 < 0$ and that $P_1'(0) = i_1(1+\Big(\tilde{\varphi}(I_1)-\tilde{\varphi}(I_2)\Big))- p_E(t) - \bar{d}_1(t) + c_{\max} \geq 0$ since by assumption $q_2(t) < q_1(t)$ therefore $\Big(\tilde{\varphi}(I_1) - \tilde{\varphi}(I_2)\Big) \geq 0$ and $c_{\max}$ needs to be greater than $p_E(t) + \bar{d}_1(t)$ since otherwise no EV driver would choose S_1 as provider. These remarks imply in turn that if polynom $P_1(.)$ admits three real roots, they are all non-negative.

In case where $\theta_1(t) > i_1$, we need to determine the $\theta_1(t)$ so that $P_1(\theta_1(t)) \geq 0$ whereas in case where $\theta_1(t) \leq i_1$, we need to determine the $\theta_1(t)$ so that $P_1(\theta_1(t)) \leq$

0. This is easily performed by comparing the positions of $\theta_1(t)$ and those of polynom $P_1(.)$'s roots.

Constraint (17) is equivalent with $(1-\theta_1(t)) + \frac{1}{\theta_1(t)-i_1}[p_E(t) + \frac{1}{\theta_1(t)} + \bar{d}_1(t) - c_{\max}] \geq 0$. To simplilfiy this inequality, we need to multiply it by $(\theta_1(t)-i_1)\theta_1(t)$.

Proceeding as above, we observe that $P_2(0) = -1$ and that $P_2'(0) = \kappa_1 - p_E(t) - \bar{d}_1(t) + c_{\max} > 0$. As cited in Constraint (16) study, two cases should be distinguished. If $\theta_1(t) > i_1$, we need to determine the $\theta_1(t)$ so that $P_2(\theta_1(t)) \leq 0$ whereas in case where $\theta_1(t) \leq i_1$, we need to determine the $\theta_1(t)$ so that $P_2(\theta_1(t)) \geq 0$. This is performed by comparing the position of $\theta_1(t)$ and those of polynom $P_2(.)$'s roots.

Depending on whether congestion occurs, the admissible area for the couple of allocations are summarized in Proposition 7 statement.

If the whole market is captured then $\theta_1(t) + \theta_2(t) = 1 \Leftrightarrow \theta_1(t) + \psi(\theta_1(t)) = 1 \Leftrightarrow p_1(t) + \bar{d}_1(t) - c_{\max} = 0$. Since at equilibrium $p_1(t) = p_E(t) + \frac{1}{\theta_1(t)}$ we infer that $\theta_1(t) = \frac{1}{c_{\max} - p_E(t) - \bar{d}_1(t)}$ when the whole market is captured.

If only a fraction $x \neq 1$ of the market is captured, solving $\theta_1(t) + \theta_2(t) = x$ is equivalent to solve a second order polynomial equation in $\theta_1(t)$: $\theta_1^2(t) + [\frac{1}{x-1}(p_E(t) + \bar{d}_1(t) - c_{\max}) + i_1]\theta_1(t) - \frac{1}{x-1} = 0$. The constant coefficient being non-negative and the polynom increasing to infinity both when $\theta_1(t) \to +\infty$ and $\theta_1(t) \to -\infty$, we infer that if the polynom admits real roots they are non-negative. Therefore, under the assumption that the parameters i_1, i_2 have been chosen so that the second-order polynom admits real roots, we conclude that the unique allocation for S_1 coincides with the smallest root whose expression is recalled in Proposition 7 statement.

$\square$

Proof of Proposition 8

Differentiating $\psi(.)$ twice with respect to $\theta_1(t)$, we obtain:
$\psi''(\theta_1(t)) = \frac{2}{(\theta_1(t)-i_1)^3}(p_E(t) + \bar{d}_1(t) - c_{\max})$. To determine the sign of $\psi''(.)$, two cases should be distinguished:

- Suppose $c_{\max} > p_E(t) + \bar{d}_1(t)$. In this case $\psi''(\theta_1(t)) < 0$ meaning that $\psi(.)$ is concave. But $\psi(0) = 1 \underbrace{- \frac{1}{i_1}[p_E(t) + \bar{d}_1(t) - c_{\max}]}_{>0} > 1$. Since $\psi(.)$ is concave, $\psi(.)$ cannot belong to the constraint space $\mathcal{C}$. Therefore, under this assumption on $c_{\max}$, System $\mathfrak{G}(3,2)$ has no solution.

- Suppose $c_{\max} \leq p_E(t) + \bar{d}_1(t)$. Provider S_1's congestion level being positive, the opportunity cost associated to provider S_1 is always greater than the maximum admissible opportunity cost. It implies that provider S_1's market share vanishes i.e., $\theta_1(t) = 0$. By substitution in $\psi(.)$ expression, we obtain provider S_2's market share: $\theta_2(t) = 1 - \frac{1}{i_1}\Big[p_E(t) + \bar{d}_1(t) - c_{\max}\Big] \leq 1$.

□

Proof of Proposition 9

By substitution of $p_1(t)$ expression in System $\mathfrak{G}(3)$, function $\psi(.)$ takes the form: $\psi(\theta_1(t)) = (1 - \theta_1(t)) + \frac{p_2(t) - \varepsilon + (\bar{d}_1(t) - c_{\max})}{\theta_1(t) - i_1}$.

We want to express $\theta_2(t)$ exclusively as a function of $\theta_1(t)$. Therefore, we multiply the equation $\theta_2(t) = \psi(\theta_1(t))$ by $\theta_1(t)$. Then, we need to solve a second-order polynomial equation in $\theta_2(t)$: $\theta_2^2(t) + \left[(\theta_1(t) - 1) - \frac{1}{\theta_1(t) - i_1}(p_E(t) + \bar{d}_1(t) + c_{\max})\right]\theta_2(t) - \frac{1}{\theta_1(t) - i_1} = 0$. Suppose that the parameter i_1 is chosen so that the polynom's discrimant remains non-negative, then $\theta_2(t)$ is obtained as the smallest or the largest root of the polynom depending whether congestion occurs. The root analytical expressions are given in Proposition 9 statement.

□

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